Phase Reduction Method for Strongly Perturbed Limit Cycle Oscillators

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The phase reduction method for limit cycle oscillators subjected to weak perturbations has significantly contributed to theoretical investigations of rhythmic phenomena. We here propose a generalized phase reduction method that is also applicable to strongly perturbed limit cycle oscillators. The fundamental assumption of our method is that the perturbations can be decomposed into a slowly varying component as compared to the amplitude relaxation time and remaining weak fluctuations. Under this assumption, we introduce a generalized phase parameterized by the slowly varying component and derive a closed equation for the generalized phase describing the oscillator dynamics. The proposed method enables us to explore a broader class of rhythmic phenomena, in which the shape and frequency of the oscillation may vary largely because of the perturbations. We illustrate our method by analyzing the synchronization dynamics of limit cycle oscillators driven by strong periodic signals. It is shown that the proposed method accurately predicts the synchronization properties of the oscillators, while the conventional method does not.

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Rhythmic phenomena are ubiquitous in nature and of great interest in many fields of science and technology, including chemical reactions, neural networks, genetic circuits, lasers, and structural vibrations [1–8]. These rhythmic phenomena often result from complex interactions among individual rhythmic elements, typically modeled as limit cycle oscillators. In analyzing such systems, the phase reduction method [1–6] has been widely used and considered an essential tool. It systematically approximates the high-dimensional dynamical equation of a perturbed limit cycle oscillator by a one-dimensional reduced *phase equation*, with just a single *phase variable* θ representing the oscillator state.

A fundamental assumption of the conventional phase reduction method is that the applied perturbation is sufficiently weak; hence, the shape and frequency of the limit cycle orbit remain almost unchanged. However, this assumption hinders the applications of the method to strongly perturbed oscillators, because the shapes and frequencies of their orbits can significantly differ from those in the unperturbed cases. Indeed, strong coupling can destabilize synchronized states of oscillators that are stable in the weak coupling limit [9]. The effect of strong coupling can further lead to nontrivial collective dynamics such as quorum-sensing transition [8], amplitude death and bistability [9], and collective chaos [10]. The assumption of weak perturbations can also be an obstacle to modeling real-world systems, which are often subjected to strong perturbations.

Although the phase reduction method has recently been extended to stochastic [11], delay-induced [12], and collective oscillations [13], these extensions are still limited to the weakly perturbed regime. To analyze a broader class of synchronization phenomena exhibited by strongly driven or interacting oscillators, the conventional theory should be extended. This Letter proposes an extension of the phase reduction method to strongly perturbed limit cycle oscillators, which enables us to derive a simple generalized phase equation that quantitatively describes their dynamics. Although not all of the above collective phenomena [8–10] are the subject of discussion in this study, our formulation will give an insight into a certain class of them, e.g., bistability between phase-locked and drifting states [9]. We use our method to analyze the synchronization dynamics of limit cycle oscillators subjected to strong periodic forcing, which cannot be treated appropriately by the conventional method.

We consider a limit cycle oscillator whose dynamics depends on a time-varying parameter $\boldsymbol{I}(t) = [I_1(t), \ldots, I_m(t)]^\top \in \mathbb{R}^m$ representing general perturbations, described by

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}(t), \boldsymbol{I}(t)), \tag{1}$$

where $\boldsymbol{X}(t) = [X_1(t), \ldots, X_n(t)]^\top \in \mathbb{R}^n$ is the oscillator state and $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) = [F_1(\boldsymbol{X}, \boldsymbol{I}), \ldots, F_n(\boldsymbol{X}, \boldsymbol{I})]^\top \in \mathbb{R}^n$ is an $\boldsymbol{I}$-dependent vector field representing the oscillator dynamics. For example, $\boldsymbol{X}$ and $\boldsymbol{I}$ can represent the state of a periodically firing neuron and the injected current, respectively [4,6]. In this Letter, we introduce a generalized phase θ, which depends on the parameter $\boldsymbol{I}(t)$, of the oscillator. In defining the phase θ, we require that the oscillator state $\boldsymbol{X}(t)$ can be accurately approximated by using $\theta(t)$ with a sufficiently small error and that $\theta(t)$ increases at a constant frequency when the parameter $\boldsymbol{I}(t)$ remains constant. The former requirement is a necessary condition for the phase reduction, i.e., for deriving a closed equation for the generalized phase, and the latter enables us to derive an analytically tractable phase equation.

To define such θ, we suppose that $\boldsymbol{I}$ is constant until further notice. We assume that Eq. (1) possesses a family of stable limit cycle solutions with period $T(\boldsymbol{I})$ and frequency $\omega(\boldsymbol{I}) := 2\pi/T(\boldsymbol{I})$ for $\boldsymbol{I} \in A$, where A is an open subset of $\mathbb{R}^m$ (e.g., an interval between two bifurcation points). An oscillator state on the limit cycle with parameter $\boldsymbol{I}$ can be parameterized by a phase $\theta \in [0, 2\pi)$ as $\boldsymbol{X}_0(\theta, \boldsymbol{I}) = [X_{0,1}(\theta, \boldsymbol{I}), \ldots, X_{0,n}(\theta, \boldsymbol{I})]^\top$. Generalizing the conventional phase reduction method [1–5], we define the phase θ such that, as the oscillator state $\boldsymbol{X}(t) = \boldsymbol{X}_0(\theta(t), \boldsymbol{I})$ evolves along the limit cycle, the corresponding phase $\theta(t)$ increases at a constant frequency $\omega(\boldsymbol{I})$ as $\dot{\theta}(t) = \omega(\boldsymbol{I})$ *for each* $\boldsymbol{I} \in A$. We assume that $\boldsymbol{X}_0(\theta, \boldsymbol{I})$ is continuously differentiable with respect to $\theta \in [0, 2\pi)$ and $\boldsymbol{I} \in A$.

We consider an extended phase space $\mathbb{R}^n \times A$, as depicted schematically in Fig. 1(a). We define $C \subset \mathbb{R}^n \times A$ as a cylinder formed by the family of limit cycles $[\boldsymbol{X}_0(\theta, \boldsymbol{I}), \boldsymbol{I}]$ for $\theta \in [0, 2\pi)$ and $\boldsymbol{I} \in A$ and define $U \subset \mathbb{R}^n \times A$ as a neighborhood of C. For each $\boldsymbol{I}$, we assume that any orbit starting from an arbitrary point $(\boldsymbol{X}, \boldsymbol{I})$ in U asymptotically converges to the limit cycle $\boldsymbol{X}_0(\theta, \boldsymbol{I})$ on C. We can then extend the definition of the phase into U, as in the conventional method [1–5], by introducing the *asymptotic phase* and *isochrons* around the limit cycle for each $\boldsymbol{I}$. Namely, we can define a generalized *phase function* $\Theta(\boldsymbol{X}, \boldsymbol{I}) \in [0, 2\pi)$ of $(\boldsymbol{X}, \boldsymbol{I}) \in U$ such that $\Theta(\boldsymbol{X}, \boldsymbol{I})$ is continuously differentiable with respect to $\boldsymbol{X}$ and $\boldsymbol{I}$, and $(\partial\Theta(\boldsymbol{X}, \boldsymbol{I})/\partial\boldsymbol{X}) \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) = \omega(\boldsymbol{I})$ holds everywhere in U, where $(\partial\Theta/\partial\boldsymbol{X}) = [(\partial\Theta/\partial X_1), \ldots, (\partial\Theta/\partial X_n)]^\top \in \mathbb{R}^n$ is the gradient of $\Theta(\boldsymbol{X}, \boldsymbol{I})$ with respect to $\boldsymbol{X}$ and the dot $(\cdot)$ denotes an inner product. This $\Theta(\boldsymbol{X}, \boldsymbol{I})$ is a straightforward generalization of the conventional asymptotic phase [1–5] and guarantees that the phase of any orbit $\boldsymbol{X}(t)$ in U always increases with a constant frequency as $\dot{\Theta}(\boldsymbol{X}(t), \boldsymbol{I}) = \omega(\boldsymbol{I})$ at each $\boldsymbol{I}$. For any oscillator state on C, $\Theta(\boldsymbol{X}_0(\theta, \boldsymbol{I}), \boldsymbol{I}) = \theta$ holds. In general, the origin of the phase can be arbitrarily defined for each $\boldsymbol{I}$ as long as it is continuously differentiable with respect to $\boldsymbol{I}$. The assumptions that $\boldsymbol{X}_0(\theta, \boldsymbol{I})$ and $\Theta(\boldsymbol{X}, \boldsymbol{I})$ are continuously differentiable can be further relaxed for a certain class of oscillators, such as those considered in Ref. [14].

Now suppose that the parameter $\boldsymbol{I}(t)$ varies with time. To define θ that approximates the oscillator state with a sufficiently small error, we assume that $\boldsymbol{I}(t)$ can be decomposed into a slowly varying component $\boldsymbol{q}(\epsilon t) \in A$ and remaining weak fluctuations $\sigma\boldsymbol{p}(t) \in \mathbb{R}^m$ as $\boldsymbol{I}(t) = \boldsymbol{q}(\epsilon t) + \sigma\boldsymbol{p}(t)$. Here, the parameters ϵ and σ are assumed to be sufficiently small so that $\boldsymbol{q}(\epsilon t)$ varies slowly as compared to the relaxation time of a perturbed orbit to the cylinder C of the limit cycles, which we assume to be $O(1)$ without loss of generality, and the oscillator state $\boldsymbol{X}(t)$ always remains in a close neighborhood of $\boldsymbol{X}_0(\theta, \boldsymbol{q}(\epsilon t))$ on C, i.e., $\boldsymbol{X}(t) = \boldsymbol{X}_0(\theta(t), \boldsymbol{q}(\epsilon t)) + O(\epsilon, \sigma)$ holds (see Supplemental Material [15]). We also assume that $\boldsymbol{q}(\epsilon t)$ is continuously differentiable with respect to $t \in \mathbb{R}$. Note that the slow component $\boldsymbol{q}(\epsilon t)$ itself does not need to be small.

Using the phase function $\Theta(\boldsymbol{X}, \boldsymbol{I})$, we introduce a generalized phase $\theta(t)$ of the limit cycle oscillator (1) as $\theta(t) = \Theta(\boldsymbol{X}(t), \boldsymbol{q}(\epsilon t))$. This definition guarantees that $\theta(t)$ increases at a constant frequency when $\boldsymbol{I}(t)$ remains

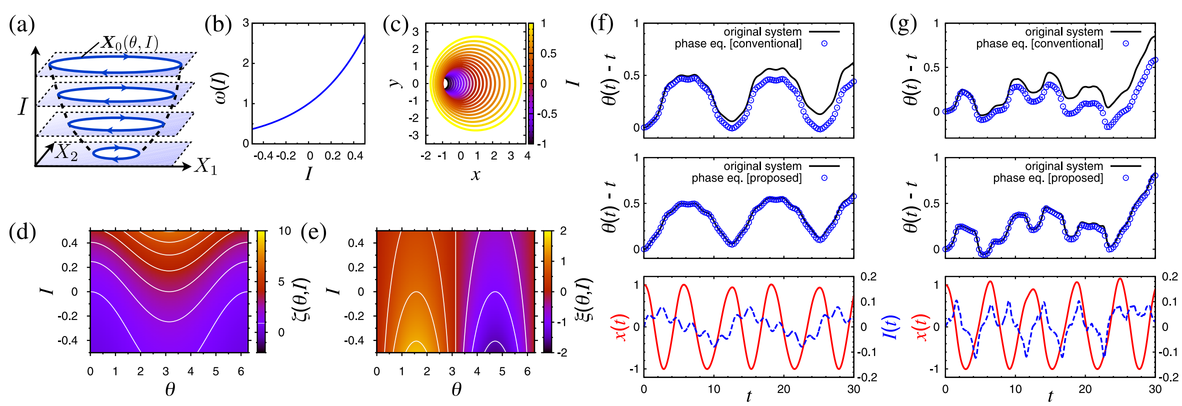

FIG. 1 (color online). Phase dynamics of a modified Stuart-Landau oscillator. (a) A schematic diagram of the extended phase space $\mathbb{R}^n \times A$ with $n = 2$ and $m = 1$. (b) Frequency $\omega(I)$. (c) I-dependent stable limit cycle solutions $\boldsymbol{X}_0(\theta, I)$. (d),(e) Sensitivity functions $\zeta(\theta, I)$ and $\xi(\theta, I)$. (f),(g) Time series of the phase $\theta(t)$ of the oscillator driven by (f) a periodically varying parameter $I^{(1)}(t)$ or (g) a chaotically varying parameter $I^{(2)}(t)$. For each of these cases, results of the conventional (top panel) and proposed (middle panel) methods are shown. Evolution of the conventional phase $\tilde{\theta}(t) = \Theta(\boldsymbol{X}(t), \boldsymbol{q}_c)$ and the generalized phase $\theta(t) = \Theta(\boldsymbol{X}(t), \boldsymbol{q}(\epsilon t))$ measured from the original system (lines) is compared with that of the conventional and generalized phase equations (circles). Time series of the state variable $x(t)$ (red solid line) and time-varying parameter $I(t)$ (blue dashed line) are also depicted (bottom panel). The periodically varying parameter is given by $I^{(1)}(t) = q^{(1)}(\epsilon t) + \sigma p^{(1)}(t)$ with $q^{(1)}(\epsilon t) = 0.05\sin(0.5t) + 0.02\sin(t)$ and $\sigma p^{(1)}(t) = 0.02\sin(3t)$, and the chaotically varying parameter is given by $I^{(2)}(t) = q^{(2)}(\epsilon t) + \sigma p^{(2)}(t)$ with $q^{(2)}(\epsilon t) = 0.007L_1(0.3t)$ and $\sigma p^{(2)}(t) = 0.001L_2(t)$, where $L_1(t)$ and $L_2(t)$ are independently generated time series of the variable x of the chaotic Lorenz equation [3], $\dot{x} = 10(y - x)$, $\dot{y} = x(28 - z) - y$, and $\dot{z} = xy - 8z/3$.

constant and leads to a closed equation for $\theta(t)$. Expanding Eq. (1) in σ as $\dot{X}(t) = F(X, q(\epsilon t)) + \sigma G(X, q(\epsilon t))p(t) + O(\sigma^2)$ and using the chain rule, we can derive $\dot{\theta}(t) = \omega(q(\epsilon t)) + \sigma(\partial\Theta(X, I)/\partial X)|_{(X(t),q(\epsilon t))} \cdot G(X, q(\epsilon t))p(t) + \epsilon(\partial\Theta(X, I)/\partial I)|_{(X(t),q(\epsilon t))} \cdot \dot{q}(\epsilon t) + O(\sigma^2)$, where $G(X, I) \in \mathbb{R}^{n\times m}$ is a matrix whose (i, j)th element is given by $\partial F_i(X, I)/\partial I_j$, $\partial\Theta/\partial I = [(\partial\Theta/\partial I_1), \ldots, (\partial\Theta/\partial I_m)]^\top \in \mathbb{R}^m$ is the gradient of $\Theta(X, I)$ with respect to I, and $\dot{q}(\epsilon t)$ denotes $dq(\epsilon t)/d(\epsilon t)$.

To obtain a closed equation for θ, we use the lowest-order approximation in σ and ϵ, i.e., $X(t) = X_0(\theta(t), q(\epsilon t)) + O(\epsilon, \sigma)$. Then, by defining a *phase sensitivity function* $Z(\theta, I) = (\partial\Theta(X, I)/\partial X)|_{(X_0(\theta,I),I)} \in \mathbb{R}^n$ and two other sensitivity functions $\zeta(\theta, I) = G^\top(X_0(\theta, I), I)Z(\theta, I) \in \mathbb{R}^m$ and $\xi(\theta, I) = (\partial\Theta(X, I)/\partial I)|_{(X_0(\theta,I),I)} \in \mathbb{R}^m$, we can obtain a closed equation for the oscillator phase $\theta(t)$ as

$$\dot{\theta}(t) = \omega(q(\epsilon t)) + \sigma\zeta(\theta, q(\epsilon t)) \cdot p(t) + \epsilon\xi(\theta, q(\epsilon t)) \cdot \dot{q}(\epsilon t) + O(\sigma^2, \epsilon^2, \sigma\epsilon), \quad (2)$$

which is a generalized phase equation that we propose in this study. The first three terms in the right-hand side of Eq. (2) represent the instantaneous frequency of the oscillator, the phase response to the weak fluctuations $\sigma p(t)$, and the phase response to deformation of the limit cycle orbit caused by the slow variation in $q(\epsilon t)$, respectively, all of which depend on the slowly varying component $q(\epsilon t)$.

To address the validity of Eq. (2) more precisely, let $\lambda(I)$ (>0) denote the absolute value of the second largest Floquet exponent of the oscillator for a fixed I, which characterizes the amplitude relaxation time scale of the oscillator $(\approx 1/\lambda(I))$. As argued in Supplemental Material [15], we can show that the error terms in Eq. (2) remain sufficiently small when $\sigma/\lambda(q(\epsilon t)) \ll 1$ and $\epsilon/\lambda(q(\epsilon t))^2 \ll 1$, namely, when the orbit of the oscillator relaxes to the cylinder C sufficiently faster than the variations in $q(\epsilon t)$.

Note that if we define the phase variable as $\tilde{\theta}(t) = \Theta(X(t), q_c)$ with some constant q_c instead of $\theta(t) = \Theta(X(t), q(\epsilon t))$, $\tilde{\theta}(t)$ gives the conventional phase. Then, we obtain the conventional phase equation $\dot{\tilde{\theta}}(t) = \omega_c + \sigma\zeta_c(\tilde{\theta}) \cdot p(t) + O(\sigma^2)$ with $q(\epsilon t) = q_c$ and $\sigma p(t) = I(t) - q_c$. Here, $\omega_c := \omega(q_c)$ is a natural frequency, $\zeta_c(\tilde{\theta}) = \zeta(\tilde{\theta}, q_c) = G(X_0(\tilde{\theta}, q_c), q_c)^\top Z(\tilde{\theta}, q_c)$, and $Z(\tilde{\theta}, q_c)$ is the conventional phase sensitivity function at $I = q_c$ [2]. This equation is valid only when $\sigma/\lambda(q_c) \ll 1$ [i.e., $\|I(t) - q_c\|/\lambda(q_c) \ll 1$]. By using the near-identity transformation [16], we can show that the conventional equation is actually a low-order approximation of the generalized equation (2) (see Sec. III of Supplemental Material [15]).

In practice, we need to calculate $\zeta(\theta, I)$ and $\xi(\theta, I)$ numerically from mathematical models or estimate them through experiments. We can show that the following relations hold (see Supplemental Material [15] for the derivation):

$$\xi(\theta, I) = -\frac{\partial X_0(\theta, I)^\top}{\partial I} Z(\theta, I), \quad (3)$$

$$\xi(\theta, I) = \xi(\theta_0, I) - \frac{1}{\omega(I)} \int_{\theta_0}^{\theta} [\zeta(\theta', I) - \bar{\zeta}(I)] d\theta', \quad (4)$$

$$\bar{\zeta}(I) := \frac{1}{2\pi} \int_0^{2\pi} \zeta(\theta, I) d\theta = \frac{d\omega(I)}{dI}, \quad (5)$$

where $(\partial X_0(\theta, I)/\partial I) \in \mathbb{R}^{n\times m}$ is a matrix whose (i, j)th element is given by $(\partial X_{0,i}(\theta, I)/\partial I_j)$, $\theta_0 \in [0, 2\pi)$ is a constant, and $\bar{\zeta}(I)$ is the average of $\xi(\theta, I)$ with respect to θ over one period of oscillation. From mathematical models of limit cycle oscillators, $Z(\theta, I)$ can be obtained numerically by the adjoint method for each I [5,6], and then $\zeta(\theta, I)$ and $\xi(\theta, I)$ can be computed from $\zeta(\theta, I) = G^\top(X_0(\theta, I), I)Z(\theta, I)$ and Eqs. (3) and (4). Experimentally, $Z(\theta, I)$ and $\zeta(\theta, I)$ can be measured by applying small impulsive perturbations to I, while $\xi(\theta, I)$ can be obtained by applying small stepwise perturbations to I.

To test the validity of the generalized phase equation (2), we introduce an analytically tractable model, a modified Stuart-Landau (MSL) oscillator (see Ref. [17] and Fig. 1 for the definition and details). We numerically predict the phase $\theta(t)$ of a strongly perturbed MSL oscillator by both conventional and generalized phase equations and compare them with direct numerical simulations of the original system. In applying the conventional phase reduction, we set $q_c = \langle I(t)\rangle_t$, where $\langle\cdot\rangle_t$ denotes the time average. In Fig. 1, we can confirm that the generalized phase equation (2) accurately predicts the generalized phase $\Theta(X(t), q(\epsilon t))$ of the original system, while the conventional phase equation does not well predict the conventional phase $\Theta(X(t), q_c)$ because of large variations in $I(t)$.

As an application of the generalized phase equation (2), we analyze k:l *phase locking* [18] of the system (1) to a periodically varying parameter $I(t)$ with period T_I and frequency ω_I, in which the frequency tuning $(l\langle\dot{\theta}\rangle_t = k\omega_I)$ occurs. Although the averaging approximation [19] for the phase difference $\tilde{\psi}(t) = l\theta(t) - k\omega_I$ is generally used to analyze the phase locking [2,18], we cannot directly apply it in the present case because the frequency $\omega(q(\epsilon t))$ can vary largely with time. Thus, generalizing the conventional definition, we introduce the phase difference as $\psi(t) = l\theta(t) - k\omega_I t - lh(t)$ with an additional term $-lh(t)$ to remove the large periodic variations in $\psi(t)$ due to $\omega(q(\epsilon t))$, where $h(t)$ is a T_I-periodic function defined as $h(t) = \int_0^t [\omega(q(\epsilon t')) - T_I^{-1} \int_0^{T_I} \omega(q(\epsilon t)) dt] dt'$. By virtue of this term, temporal variations in ψ remain of the order $O(\epsilon, \sigma)$, i.e., $|\dot{\psi}| \ll 1$, which enables us to apply the averaging approximation to ψ.

Introducing a small parameter ν representing the magnitude of variations in ψ, one can derive a dynamical equation for ψ

as $\dot{\psi}(t) = \nu f(\psi, t)$, where $\nu f(\psi, t) = l g(\psi/l + k\omega_I t/l + h(t), t) - k\omega_I - l\dot{h}(t)$ and $g(\theta, t)$ denotes the right-hand side of Eq. (2). Using first- and second-order averaging [19], we can introduce slightly deformed phase differences $\psi_{1,2}$ satisfying $\psi_{1,2}(t) = \psi(t) + O(\nu)$ and obtain the first- and second-order averaged equations $\dot{\psi}_1(t) = \nu \bar{f}_1(\psi_1) + O(\nu^2)$ and $\dot{\psi}_2(t) = \nu \bar{f}_1(\psi_2) + \nu^2 \bar{f}_2(\psi_2) + O(\nu^3)$, where $\bar{f}_1(\psi)$ and $\bar{f}_2(\psi)$ are given by $\bar{f}_1(\psi) = (lT_I)^{-1} \times \int_0^{lT_I} f(\psi, t)dt$ and $\bar{f}_2(\psi) = (lT_I)^{-1} \int_0^{lT_I}[u(\psi, t) \times (\partial f(\psi, t)/\partial \psi) - \bar{f}_1(\psi)(\partial u(\psi, t)/\partial \psi)]dt$, respectively, and $u(\psi, t) = \int_0^t [f(\psi, t') - \bar{f}_1(\psi)]dt'$. These averaged equations can be considered autonomous by neglecting the $O(\nu^2)$ and $O(\nu^3)$ terms, respectively. Averaged equations for the conventional phase equation can be derived similarly. Thus, if the averaged equation has a stable fixed point, k:l phase locking is expected to occur. As demonstrated below, the first-order averaging of the generalized phase equation already predicts qualitative features of the phase-locking dynamics, while the second-order averaging gives more precise results when the parameter $\boldsymbol{I}(t)$ varies significantly.

As an example, we use the MSL oscillator and investigate their phase locking to periodic forcing (See Supplemental Material [15] for other examples). Figure 2 shows the results of the numerical simulations. We apply four types of periodically varying parameters and predict if the oscillator exhibits either 1:1 or 1:2 phase locking to the periodically varying parameter $q(\epsilon t)$ [a small fluctuation $\sigma p(t)$ is also added for completeness]. We derive averaged equations for the phase differences $\psi_{1,2}$ by using the proposed and conventional methods and compare the results with direct numerical simulations of the MSL oscillator. We find that our new method correctly predicts the stable phase-locking point already at first-order averaging, while the conventional method does not. In particular, the conventional method can fail to predict whether phase locking takes place or not, as shown in Figs. 2(g) and 2(h), even after the second-order averaging. In this case, the exponential dependence of the frequency $\omega(I)$ on the parameter I is the main cause of the breakdown of the conventional method (see Sec. III of Supplemental Material [15] for a discussion). Typical trajectories of $[x(t), y(t), q(\epsilon t)]^\top$ are plotted on the cylinder C of limit cycles in the extended phase space $[x, y, I]^\top$, which shows that the oscillator state migrates over C synchronously with the periodic forcing. The trajectories are closed when phase locking occurs.

In summary, we proposed a generalized phase reduction method that enables us to theoretically explore a broader class of strongly perturbed limit cycle oscillators. Although still limited to slowly varying perturbations with weak

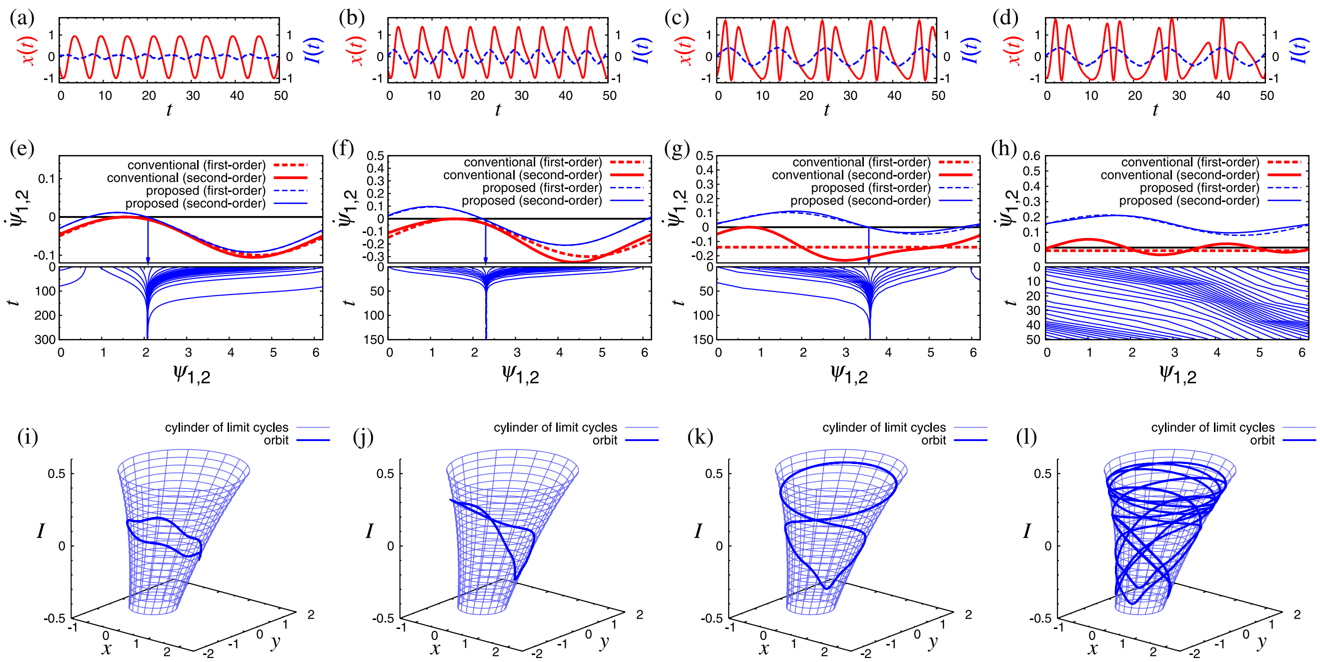

FIG. 2 (color online). Phase locking of the modified Stuart-Landau oscillator. Four types of periodically varying parameters $I^{(j)}$ ($j = 3, 4, 5, 6$) are applied, which lead to 1:1 phase locking to $I^{(3)}(t)$ [(a), (e), and (i)], 1:1 phase locking to $I^{(4)}(t)$ [(b), (f), and (j)], 1:2 phase locking to $I^{(5)}(t)$ [(c), (g), and (k)], and failure of phase locking to $I^{(6)}(t)$ [(d), (h), and (l)]. (a)–(d) Time series of the state variable $x(t)$ of a periodically driven oscillator (red solid line) and periodic external forcing (blue dashed line). (e)–(h) Dynamics of the phase difference $\psi_{1,2}$ with an arrow representing a stable fixed point (top panel) and time series of $\psi_{1,2}$ with 20 different initial states (bottom panel). (i)–(l) Orbits of a periodically driven oscillator (blue line) on the cylinder of the limit cycles (light blue line) plotted in the extended phase space. The parameter $I^{(j)}(t)$ is given by $I^{(j)}(t) = q^{(j)}(\epsilon t) + \sigma p^{(j)}(t)$, $q^{(j)}(\epsilon t) = \alpha^{(j)} \sin(\omega_I^{(j)} t)$, and $\sigma p^{(j)}(t) = 0.02 \sin(5\omega_I^{(j)} t)$ with $\alpha^{(3,4,5,6)} = 0.1, 0.3, 0.4, 0.4$ and $\omega_I^{(3,4,5,6)} = 1.05, 1.10, 0.57, 0.51$.

fluctuations, our method avoids the assumption of weak perturbations, which has been a major obstacle in applying the conventional phase reduction method to real-world phenomena. It will therefore facilitate further theoretical investigations of nontrivial synchronization phenomena of strongly perturbed limit cycle oscillators [9,10]. As a final remark, we point out that a phase equation similar to Eq. (2) has been postulated in a completely different context, to analyze the *geometric phase* in dissipative dynamical systems [20]. This formal similarity may provide an interesting possibility of understanding synchronization dynamics of strongly perturbed oscillators from a geometrical viewpoint.

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