NOLTA2006
11-14september
Bologna-ITALY

Reproducibility of limit-cycle oscillators induced by random impulses

Kensuke Arai and Hiroya Nakao

Department of Physics, Kyoto University,
Kyoto 606-8502, Japan
Email: arai@ton.scphys.kyoto-u.ac.jp, nakao@ton.scphys.kyoto-u.ac.jp

Abstract—
We argue that the reliability of a limit-cycle oscillator generally improves when it is driven by a random impulsive input. By reducing the dynamics of the oscillator to a stochastic phase equation, we argue generally that the orbit of a limit-cycle oscillator is statistically stabilized against phase disturbances when it is driven by a weak impulsive input regardless of its details, leading to an improvement in its reproducibility. We demonstrate our theoretical results by numerical simulations and experiments with a simple electrical oscillator.

1. Introduction

It is now well established that mutually interacting limit-cycle oscillators synchronize with each other [1]. Recently, it has been demonstrated that nonlinear oscillators can synchronize with each other through common fluctuating inputs, even in the absence of direct mutual interactions. Such noise-induced synchronization of an ensemble of nonlinear oscillators can also be interpreted as noise-induced improvement in the reproducibility of the orbit of a single oscillator, because repeated experiments with a single oscillator using the same input signal is equivalent to a single experiment with multiple oscillators using a common input signal. Noise-induced synchronization, reproducibility, or "consistency" as termed by some authors, is widely observed in experimental systems ranging from lasers to neurons [2, 3], where the dynamics of each element can also be chaotic or stochastic.

For example, in electrophysiological experiments using a single neuron from slice preparations of rat neocortex or olfactory bulb [3], it is observed that the neuron evokes different spike sequences from trial to trial when it is driven by a constant input current, whereas it evokes mostly the same spike sequences over the trials when it is driven by a fluctuating input current. This phenomenon can be interpreted as a noise-induced improvement in the reproducibility of the orbit of a limit-cycle oscillator induced by a fluctuating input signal.

On the theoretical aspect, after several pioneering works [4], Teramae & Tanaka and Goldobin & Pikovsky [5] adopted the phase reduction method [1] and proved generally that uncoupled limit-cycle oscillators always synchronize with each other when they are driven by a common weak Gaussian white noise. Using a similar idea, we also argued generally that uncoupled limit-cycle oscillators can synchronize with each other when they are driven by random impulsive, telegraphic, or piecewise-constant input signals [6].

In this paper, we adopt the theory of Poisson driven Markov process [7] to analyze the case of impulsive input signals. We generally prove that the reproducibility of an oscillator orbit always improves for sufficiently weak Poisson impulses. We also argue that when the impulses are not weak, the oscillators can also undergo desynchronization rather than synchronization. We demonstrate our theoretical results experimentally by observing the effects of Poisson impulses applied to a simple electrical circuit exhibiting limit-cycle oscillations.

2. Numerical Example

As an example, we first present the result of direct numerical simulations using the FitzHugh-Nagumo (FN) neural oscillator. The FN oscillator is described by

$$\begin{aligned} \dot{u}(t) &= \varepsilon(v + c - du), \\ \dot{v}(t) &= v - v^3/3 - u + I_0 + I(t) + \xi(t), \end{aligned} \qquad (1)$$

where $v(t)$ represents the membrane potential of the neuron at time t, $u(t)$ the state of the ion channels in a reduced form, ε, c, and d are parameters. I_0 and $I(t)$ represent the constant component and the fluctuating component of the input current. $\xi(t)$ represents a weak Gaussian-white noise specified by $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = D\delta(t-s)$, which incorporates the effect of various fluctuations. When $\xi(t)$ and $I(t)$ are absent and I_0 is fixed at a constant value in some appropriate range, this model exhibits a typical limit-cycle oscillation, which corresponds to the periodic spiking of the neuron. We fix the parameters at $\varepsilon = 0.08$, $c = 0.7$, $d = 0.8$, $I_0 = 0.8$, and $D = 0.0001$ in the following.

Figure 1 displays the result of 50 repeated numerical simulations of Eq. (1), where the spiking times of the oscillator (the point at which v changes its sign from negative to positive) are plotted by dots. In Fig. 1(a), the oscillator is driven only by the constant input current I_0. Due to the weak noise $\xi(t)$ applied independently at every trial, the spiking times of the oscillators are considerably scattered. This is due to the neutral stability of the limit cycle in the phase direction. The orbital component of the perturbations do not decay and gradually accumulate, which results

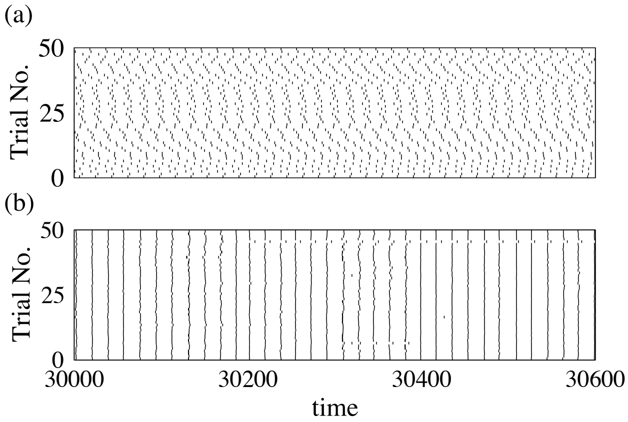

Figure 1: 50 spiking sequences of the FN oscillator subject to (a) only constant input current, and (b) random impulses in addition to the constant input current.

in phase diffusion. Figure 1(b) displays the result of 50 repeated simulations in which the oscillators are driven not only by the constant current I_0 but also by random Poisson impulses $I(t)$ given by

$$I(t) = \sum_{n=1}^{N(t)} I_n \delta(t - t_n), \tag{2}$$

where $\{t_1, t_2, \cdots\}$ represent the generation times of the impulses, and $\{I_1, I_2, \cdots\}$ the intensity of the impulses. We set the mean interval between the impulses at $\tau = 10$ and assume that the intensity I_n of the impulse takes ± 0.5 with equal probability. In this case, the spiking times of the oscillator are reliably reproduced over the trials after an initial transient, even under the effect of the weak independent noises. We analyze the mechanism leading to such behavior in the next section.

3. Phase Reduction Analysis

The improvement in reproducibility is due to the statistical stabilization of perturbations in the phase direction induced by random impulses. We formulate this fact using stochastic differential equation in this section. Our model is generally described by the following random dynamical system:

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}; \boldsymbol{I}_0) + \boldsymbol{I}(t), \quad \boldsymbol{I}(t) = \sum_{n=1}^{N(t)} \boldsymbol{I}_n \delta(t - t_n), \tag{3}$$

where $\boldsymbol{X}$ represents the dynamical variable of the oscillator, $\boldsymbol{F}$ its dynamics, $\boldsymbol{I}_0$ the constant component of the external input, and $\boldsymbol{I}(t)$ the random impulsive input. We assume that the impulses are generated by a Poisson process of rate λ (so that the mean interval is $\tau = \lambda^{-1}$). We denote the total number of impulses generated up to time t by $N(t)$, and the generation times and intensities of the n-th impulse by $\{t_n, \boldsymbol{I}_n\}$. Each impulse intensity $\boldsymbol{I}_n$ is chosen randomly from a probability density function $Q(\boldsymbol{I})$. We omit the weak independent noise in the following analysis, which is not important in the linear stability analysis. We assume that the system has a single limit cycle $\boldsymbol{X}_0(t)$ when only the constant input $\boldsymbol{I}_0$ is given, and also that most initial conditions in the phase space are eventually attracted to this limit cycle.

When λ is small, the mean interval between the impulses becomes long, so that the oscillator almost always receives only the constant input $\boldsymbol{I}_0$. The orbit is kicked off from the limit cycle occasionally by an impulse, but it returns to the limit cycle sufficiently quickly before the arrival of the next impulse. In such a situation, we can reduce the evolution equation (3), which generally contains multiple variables, to a simplified equation of the scalar phase variable only. In our previous paper [6], we formulated such phase reduction using random phase maps. Here, we adopt the methods of stochastic differential equations [7] for this purpose.

We first rewrite Eq. (3) as a stochastic differential equation driven by the Poisson impulses as [7]

$$d\boldsymbol{X}(t) = \boldsymbol{F}(\boldsymbol{X}; \boldsymbol{I}_0)dt + \int \boldsymbol{I} M(dt, d\boldsymbol{I}). \tag{4}$$

Here, $M(dt, d\boldsymbol{I})$ denotes the Poisson random measure [7], which represents the number of impulses generated in the time interval $[t, t + dt]$ and whose intensity is in the range $[\boldsymbol{I}, \boldsymbol{I} + d\boldsymbol{I}]$. Its expectation value is given by $E[M(dt, d\boldsymbol{I})] = \lambda Q(\boldsymbol{I}) dt d\boldsymbol{I}$.

We then introduce a phase variable $\theta(\boldsymbol{X}) \in [0, 1]$ along the limit cycle $\boldsymbol{X}_0(t)$ corresponding to the constant input $\boldsymbol{I}_0$, which increases with constant angular velocity ω. We can extend this definition of the phase variable to the whole of phase-space (except phase singular points) by assigning the same phase value to the set of points that eventually converge to the same point on the limit cycle [1].

By applying the (generalized) Ito formula for the Poisson driven Markov process [7] to Eq. (4), we obtain

$$d\theta(t) = \omega dt + \int \left[\theta(\boldsymbol{X}(t) + \boldsymbol{I}) - \theta(\boldsymbol{X}(t))\right] M(dt, d\boldsymbol{I}). \tag{5}$$

Furthermore, by assuming that the orbit is (almost) always on the limit cycle when it receives an impulse, we approximate the $\boldsymbol{X}(t)$ in the above equations by $\boldsymbol{X}_0(\theta(t))$. We then obtain a closed phase equation for $\theta(t)$,

$$d\theta(t) = \omega dt + \int G(\theta(t), \boldsymbol{I}) M(dt, d\boldsymbol{I}), \tag{6}$$

where the function $G(\theta, \boldsymbol{I})$ is defined as

$$G(\theta, \boldsymbol{I}) = \theta(\boldsymbol{X}_0(\theta) + \boldsymbol{I}) - \theta. \tag{7}$$

$G(\theta, \boldsymbol{I})$ is a periodic function in θ representing the change in the phase of the orbit after receiving an impulse of intensity $\boldsymbol{I}$ at the point $\boldsymbol{X}_0(\theta)$ on the limit cycle. We refer to this function as "phase map" hereafter.

Now we consider the statistical stability of the phase against small perturbations. A linearized equation for the evolution of a small perturbation $\psi(t)$ to the original phase $\theta(t)$ is obtained from Eq. (6) as

$$d\psi(t) = \int G'(\theta(t), \boldsymbol{I})\psi(t)M(dt, d\boldsymbol{I}), \tag{8}$$

where $'$ denotes differential by θ. By applying the Ito formula to this equation, we obtain the following equation for the logarithm of the absolute phase perturbation $\ln|\psi(t)|$:

$$d \ln|\psi(t)| = \int \ln\left|1 + G'(\theta(t), \boldsymbol{I})\right| M(dt, d\boldsymbol{I}). \tag{9}$$

By taking the expectation, we obtain

$$E[d\ln|\psi(t)|] = \Lambda dt, \tag{10}$$

where we defined the Lyapunov exponent Λ that quantifies the mean growth rate of the small perturbation as

$$\begin{aligned}\Lambda &= E[\ln\left|1 + G'(\theta(t), \boldsymbol{I})\right|] \\ &= \lambda\int d\boldsymbol{I}Q(\boldsymbol{I})\int d\theta P(\theta)\ln\left|1 + G'(\theta, \boldsymbol{I})\right|. \end{aligned} \tag{11}$$

Here, $P(\theta)$ is the stationary probability density of the phase θ. When this Λ is negative, the phase perturbation decays exponentially, resulting in the improvement of reproducibility of the limit-cycle oscillator.

Particularly, when the intensities of the external impulses are sufficiently small, we can generally argue for the negativity of Λ as follows. In this case, $G(\theta, \boldsymbol{I})$ can be approximated as

$$G(\theta, \boldsymbol{I}) \simeq \boldsymbol{Z}(\theta)\cdot\boldsymbol{I}, \tag{12}$$

where $\boldsymbol{Z}(\theta) := \nabla_{\boldsymbol{X}}\theta(\boldsymbol{X})|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta)}$ is the conventional phase sensitivity function [1]. Since $\boldsymbol{Z}(\theta)$ is a smooth periodic function in θ, $G(\theta, \boldsymbol{I})$ is also smooth and does not fluctuate largely when $|\boldsymbol{I}|$ is small. In such a situation, $1 + G'(\theta, \boldsymbol{I})$ is always positive. Also, it is physically apparent that the probability density function of the phase $\theta(t)$ becomes almost uniform,

$$P(\theta) \simeq 1, \tag{13}$$

when the impulses are weak (this can also be shown analytically using the Master or the Frobenius-Perron equations). Now, by using the periodicity of $G(\theta, \boldsymbol{I})$ in θ, we can generally prove the negativity of Λ as follows:

$$\begin{aligned}\Lambda &\simeq \lambda\int d\boldsymbol{I}Q(\boldsymbol{I})\int d\theta\ln\left|1 + G'(\theta, \boldsymbol{I})\right| \\ &\leq \lambda\int d\boldsymbol{I}Q(\boldsymbol{I})\int d\theta G'(\theta, \boldsymbol{I}) \\ &= \lambda\int d\boldsymbol{I}Q(\boldsymbol{I})\{G(1, \boldsymbol{I}) - G(0, \boldsymbol{I})\} \\ &= 0. \end{aligned} \tag{14}$$

The equality holds only in the trivial case of the constant $G(\theta, \boldsymbol{I})$. Thus, we can prove that Λ is always less than 0 for a general class of limit-cycle oscillators regardless of their details when the random impulses are sufficiently weak. Therefore, it is expected that the reproducibility of a limit-cycle oscillator always improves when it is driven by sufficiently weak random impulses. In the next section, we will demonstrate this fact using an electric circuit receiving random impulses.

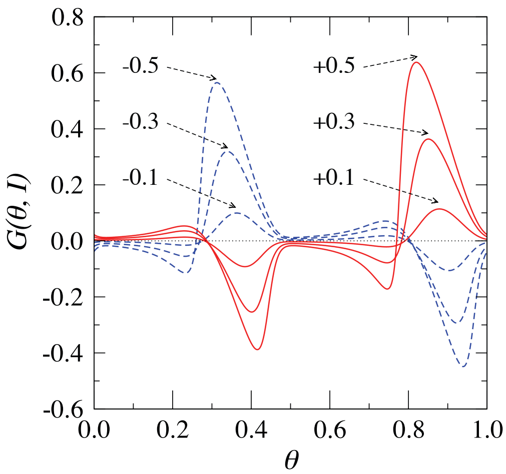

Figure 2: Phase maps of the FN model for several values of the impulse intensity I.

Let us examine the phase maps of the FN oscillator now. Figure 2 displays the phase maps $G(\theta, I)$ obtained for several values of the impulse intensity I. When $|I|$ is small, the amplitude of the corresponding phase maps are also small, so that Λ is negative from the above argument. Thus, the reproducibility improves by the external impulses as demonstrated in Fig. 1(b). In contrast, when $|I|$ is large, the amplitude of the corresponding phase maps can also become large and fluctuates strongly (Eq.(12) does not hold for large impulses). Λ can be positive in such cases, leading to impulse-induced desynchronization.

4. Electric Circuit Experiments

In this section, we demonstrate the phenomenon of impulse-induced reproducibility in a real experimental system using a very simple electric circuit.

Our electric circuit was originally designed for flashing an LED periodically (Fig. 3(a)). We use a computer equipped with an AD/DA-converter board (CONTEC ADA16-8/2) to generate an output signal (the impulses), and to measure the voltages V_1 and V_2 at 2 separate locations in the circuit. Using a MOSFET, voltage impulses from the AD/DA board was used to briefly short circuit a given section of the oscillator. The strength of the impulse delivered was varied by adding a variable resistor in series with the MOSFET. Changing the location in the circuit in which to deliver the impulse naturally changes the response of the circuit to a given impulse.

Figure 3(b) displays typical limit cycle of the electric circuit, which is qualitatively similar to the FN oscillator we treated above. Figure 3(c) shows experimentally observed phase maps obtained for two different intensities of the impulses (i.e. the magnitude of the series resistor). With these intensities of the impulses, the function $G(\theta, \mathbf{I})$ is sufficiently smooth, and the absolute value of $G'(\theta, \boldsymbol{I})$ is always smaller than 1. Therefore, we anticipate impulse-induced reproducibility in this case. The results are shown in Figs. 3(d) and (e), where two time sequences of V_1 measured in two different experimental trials, and and their spiking times are plotted. Under the effect of random impulses, the two temporal sequences behave quite similarly. In contrast, without impulses, the two temporal sequences behave quite differently due to the internal noises of the electric circuit. Thus, we see that noise-induced synchronization can easily be observed in our simple experiment.

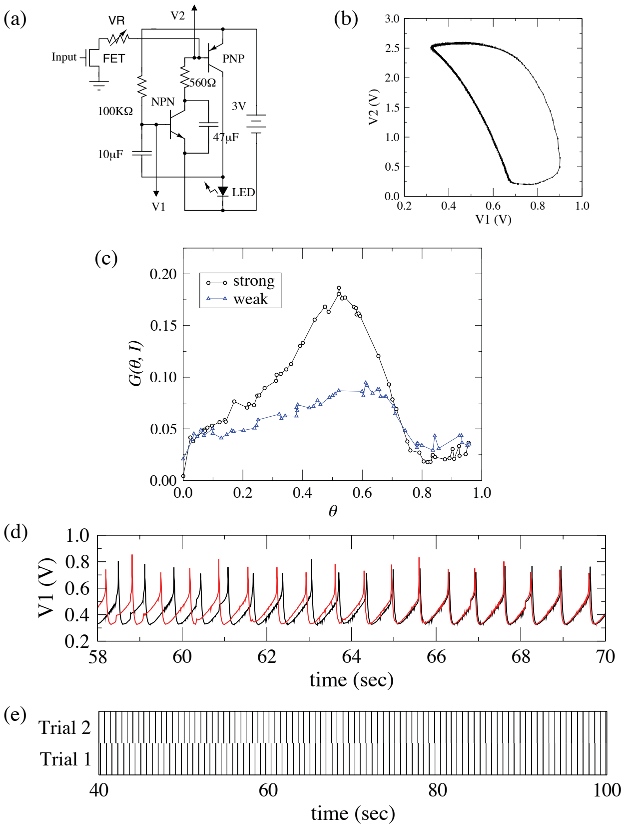

Figure 3: Results of the circuit experiment. (a) circuit diagram, (b) limit cycle, (c) phase maps $G(\theta)$ obtained using two intensities of the impulses, (d) superposition of two measured temporal sequences of $V1$, and (e) spiking times.

Though noise-induced synchronization has already been observed in a wide class of systems, we believe that carefully controlled experiments using simple experimental systems can also be useful in deepening our insight in this phenomenon. Specifically, the experimental measurement of the phase map, which is rather difficult in more complicated and widely fluctuating systems such as neurons, can be accomplished simply and unambiguously in a simple system such as this. Our results reported here are only preliminarily. More detailed quantitative analysis will be reported in the future.

5. Summary

We proved theoretically that the reproducibility of noisy limit-cycle oscillators generally improves when driven by random impulses, and experimentally confirmed this theoretical prediction using an electric circuit. Our theory holds for any limit-cycle oscillators that satisfy our assumptions irrespective of their detailed structures. We thus expect that synchronization or reproducibility of randomly driven nonlinear oscillators caused by this mechanism can be observed in various natural phenomena.

Acknowledgments

The authors thank Y. Kuramoto, J. Teramae, Y. Tsubo, K. Nagai, and A. Uchida for useful comments.

References

[1] A. T. Winfree, *The Geometry of Biological Time* (Springer, 1980); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Dover, 2003); A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge, 2001).

[2] A. Uchida, R. McAllister, and R. Roy, Phys. Rev. Lett. **93**, 244102 (2004).

[3] Z. F. Mainen and T. J. Sejnowski, Science **268** (1995) 1503; R. F. Galán, N. Fourcaud-Trocmé, G. B. Ermentrout, and N. N. Urban, Journal of Neuroscience **26**, 3646 (2006).

[4] R. V. Jensen, Phys. Rev. E. **58**, R6907 (1998); K. Pakdaman, Neural Comput. **14**, 781 (2002); B. Gutkin, G. B. Ermentrout, and M. Rudolph, J. Comput. Neurosci. **15**, 91 (2003); J. Ritt, Phys. Rev. E **68**, 041915 (2003).

[5] J. Teramae and D. Tanaka, Phys. Rev. Lett. **93** (2004) 204103; D. S. Goldobin and A. Pikovsky, Phys. Rev. E **71**, 045201(R) (2005).

[6] K. Nagai, H. Nakao, and Y. Tsubo, Phys. Rev. E **71**, 036217 (2005); H. Nakao, K. Arai, K. Nagai, Y. Tsubo, and Y. Kuramoto, Phys. Rev. E **72**, 026220 (2005); H. Nakao, K. Nagai, and K. Arai, Prog. Theoret. Phys. Suppl. **161**, 294 (2006).

[7] I. Gihman and A. Skorohod, *Introduction to the Theory of Random Processes* (Saunders, 1969); D. L. Snyder, *Random Point Processes* (John Wiley & Sons, 1975).